Handout

Hi! You can take a handout, if there is one.

The Renormalization Group (RG)

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Outline

RG as a set of symmetry transformations

- Given: macroscopic physical system near a **critical point**.
- RG for our system: set of *symmetry transformations* (called **RG transformations**).
- idea of these symmetry transformations: $\sim$ same as in Atomic physics (for example). **Differences:**
  - The whole system will not be simply invariant under a RG transformation.
  - It is not so easy to *find* a RG.
  - The RG is *not* a group.
- carrying out a symmetry transformation means not the same as solving a particular model

Our Job

- **Our job:** Find a RG for our system (there may be many).
- **The idea:** Study the behavior of our system under a change of scale.

What do we get from RG theory?

- We get a **connection to QFT**.
- The **critical exponents appear as symmetry properties** of the RG transformations:
  - As a direct consequence of having found a RG for our system we get the **scaling relations** corresponding to our system.
  - But not only the scaling relations: there is a way to **calculate critical exponents**.

Framework

- **Theory of Phase Transitions**
- **Theory of Critical Phenomena**
- **Classical statistical mechanics**
- **Lattice spin systems**

Bibliography

This talk is mainly based on a paper of J. Fröhlich and T. Spencer, see [1] in Bibliography (Handout).

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   - Framework, main idea
   - Classical lattice spin systems

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   - Kadanoff (RG) block spin transformations
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Justifications

In the following we will only consider the simplest, classical spin systems, namely lattice ones. For the following treatment we should justify the following:

- the use of **classical** spin systems.
- considering **infinitely large** lattice systems.
- considering only the **(hyper-) cubic** lattice.

1. Lattice

The lattice is chosen to be the simple **(hyper-) cubic lattice**, $\mathbb{Z}^{\mathbf{d}}$. With each lattice site $j \in \mathbb{Z}^d$ associated is a **classical spin**

$$\varphi(j) \in \mathbb{R}^N_{(j)} \cong \mathbb{R}^N ,$$

where N is the dimension of spin space (for all lattice sites the same). For $\Lambda \subset \mathbb{Z}^d$ define the **spin configuration on** Λ to be

$$\varphi_\Lambda := \{\varphi(j) : j \in \Lambda\} ,$$

which is contained in the **space of all spin configurations on** Λ,

$$K_\Lambda := \prod_{j \in \Lambda} \mathbb{R}^N_{(j)} .$$

We set $K_\infty \equiv K_{\mathbb{Z}^d}$.

2. Spin

The further characterization of the spin is given by the **a priori distribution of the spin.** For an arbitrary spin $\varphi(j)$, $j \in \mathbb{Z}^d$, it is given by a probability measure, $d\lambda(\varphi(j))$, on the Borel sets of $\mathbb{R}^N$. Note that it is defined to be the same for all sites $j \in \mathbb{Z}^d$.
Now let $\Lambda \subset \mathbb{Z}^d$ be a *finite* subset of the lattice. Then the a priori distribution of a spin configuration on Λ is given by

$$\prod_{j \in \Lambda} d\lambda(\varphi(j)) ,$$

which is a probability measure on K_Λ.

3. Energy

- For a finite sublattice Λ we define the **Hamilton function**:

$$H_\Lambda : K_\Lambda \to \mathbb{R}\,,\ \varphi_\Lambda \mapsto H_\Lambda(\varphi_\Lambda)$$

- Assumption: $H_\Lambda(\varphi_\Lambda)$ is *continuous* on K_Λ.
- $\beta \equiv (kT)^{-1} \geq 0$: **inverse temperature**

$(\Lambda_n)_{n\in\mathbb{N}}$: arbitrary sequence of finite regions in $\mathbb{Z}^d$ increasing to $\mathbb{Z}^d$.
Assumption: the following thermodynamic limits exist:

- W_{Λ,Λ^c}: **interaction energy** between the spins in the finite region Λ and the ones outside Λ

$$W_{\Lambda,\Lambda^c} \equiv \lim_{n\to\infty} W_{\Lambda,\Lambda_n\setminus\Lambda} \equiv \lim_{n\to\infty} \{H_{\Lambda_n} - (H_\Lambda + H_{\Lambda_n\setminus\Lambda})\} \quad , \quad \forall \text{ finite } \Lambda \, .$$

- $f(\beta,\lambda)$: **free energy per unit volume**

$$\beta f(\beta,\lambda) \equiv \lim_{n\to\infty} -\frac{1}{|\Lambda_n|} \log Z_\beta(\Lambda_n) \qquad , \quad \forall\beta \geq 0 \quad ,$$

where

$$Z_\beta(\Lambda_n) \equiv \int_{K_{\Lambda_n}} exp[-\beta H_{\Lambda_n}(\varphi_{\Lambda_n})] \prod_{j\in\Lambda_n} d\lambda(\varphi(j)) \, .$$

4. Equilibrium state

An **equilibrium state at inverse temperature** β of the *infinite* lattice spin system is given by a probability measure, $d\mu_{\beta,\lambda}(\varphi)$, on K_∞ with the following *defining property*:
For every bounded measurable function A on K_Λ, where Λ is an *arbitrary* finite sublattice, the following holds:

$$\begin{aligned}\langle A\rangle_{\beta,\lambda} &\equiv \int_{K_\infty} A(\varphi_\Lambda)\, d\mu_{\beta,\lambda}(\varphi) \\ &= \int_{K_{\Lambda^c}} d\rho(\varphi_{\Lambda^c}) \int_{K_\Lambda} e^{-\beta\left(W_{\Lambda,\Lambda^c}(\varphi)\,+\,H_\Lambda(\varphi_\Lambda)\right)} A(\varphi_\Lambda) \prod_{j\in\Lambda} d\lambda(\varphi(j))\,,\end{aligned}$$

where $d\rho(\varphi_{\Lambda^c})$ is a finite measure on K_{Λ^c}. These are the so called **Dobrushin-Lanford-Ruelle equations**.

Simplification

For simplicity: $N = 1$ (N: dimension of spin space)

Correlation functions

$$\langle\, \varphi(x_1)\ \cdots\ \varphi(x_n)\,\rangle_\beta \equiv \left\langle \prod_{k=1}^{n} \varphi(x_k) \right\rangle_\beta = \int_{K_\infty} \prod_{k=1}^{n} \varphi(x_k)\ d\mu_\beta(\varphi)\,. \tag{1}$$

Inverse correlation length

Of particular importance in the following are

- the **inverse correlation length** (mass)

$$m(\beta) := \xi^{-1}(\beta) := -\lim_{|x|\to\infty} \frac{1}{|x|} \log \langle \varphi(0)\varphi(x)\rangle_\beta^c \,, \tag{2}$$

  which measures the exponential decay rate of $\langle \varphi(0)\varphi(x)\rangle_\beta^c$, and
- the **susceptibility**

$$\chi(\beta) := \sum_{x\in\mathbb{Z}^d} \langle \varphi(0)\varphi(x)\rangle_\beta^c \,, \tag{3}$$

where

$$\langle \varphi(0)\varphi(x)\rangle_\beta^c := \langle \varphi(0)\varphi(x)\rangle_\beta - \langle \varphi(0)\rangle_\beta^2 \,. \tag{4}$$

Critical point

In the following we will only consider phase transitions at a critical point. $\beta = \beta_c$ is defined to be a **critical point** if

$$m(\beta) \searrow 0 \quad , \quad \text{as } \beta \nearrow \beta_c \quad , \quad \text{or } \beta \searrow \beta_c \, . \tag{5}$$

Scaling laws

For the two above defined quantities, the *inverse correlation length* and the *susceptibility*, we expect that the corresponding **scaling laws**

$$\tilde{m}(t) \sim t^{\nu} \quad , \quad \text{as} \ \ t \to 0 \tag{6}$$

$$\tilde{\chi}(t) \sim t^{-\gamma} \quad , \quad \text{as} \ \ t \to 0 \tag{7}$$

hold, where $\tilde{m}(t) := m(\beta)$, $\tilde{\chi}(t) := \chi(\beta)$, $t := \frac{\beta_c - \beta}{\beta_c}$ and

$$a(t) \sim b(t) \ , \ t \to 0 \qquad :\Leftrightarrow \qquad \lim_{t \to 0+} \frac{a(t)}{b(t)} = 1 \ . \tag{8}$$

Outline

For the following we consider the case in which

$$\beta < \beta_c \; .$$

Assumptions (equilibrium state, inv. correl. length):

- The equilibrium state $d\mu_\beta(\varphi)$ is *invariant under lattice translations* , i.e.

$$\forall x \in \mathbb{Z}^d : \quad d\mu_\beta(\varphi_x) = d\mu_\beta(\varphi) \; ,$$

  where

$$\varphi_x(j) = \varphi(j + x) \; , \qquad x, j \in \mathbb{Z}^d \; .$$

- For $\beta < \beta_c$, the equilibrium state $d\mu_\beta$ is *extremal invariant*, i.e. ergodic under the action of lattice translations.
- the inverse correlation length, $m(\beta)$, is positive and continuous in β, with

$$m(\beta) \searrow 0 \quad , \quad \text{as} \;\; \beta \nearrow \beta_c \; . \tag{9}$$

Large scale behavior of the correlation functions

Due to the divergence of the correlation length, $\xi \equiv m(\beta)^{-1}$, at a critical point we are interested in the **large scale behavior of the correlation functions**

$$\langle \varphi(z_1) \cdots \varphi(z_n) \rangle_\beta \quad , \quad \beta \text{ near to } \beta_c \qquad (z_i \in \mathbb{Z}^d, 1 \leq i \leq n) \; ; \tag{10}$$

more precisely we are interested in the long distance limit of (10), in which

$$|z_i - z_j| \to \infty \quad , \quad \text{whenever } i \neq j \, . \tag{11}$$

Scale transformation

In order to realize such a limit we consider a **scale transformation**

$$\mathbb{Z}^d \to \mathbb{Z}^d_{\vartheta^{-1}} := \{y \in \mathbb{R}^d \ : \ \vartheta y \in \mathbb{Z}^d\} \quad , \qquad z \mapsto \vartheta^{-1} z \tag{12}$$

where $\vartheta \in [1, \infty[$ is called a **scale parameter.** Next, for given points $x_k \in \mathbb{Z}^d \subset \mathbb{R}^d$, $1 \leq k \leq n$, we choose for every $\vartheta \in [1, \infty[$ functions $\vartheta \mapsto x_i(\vartheta) \in \mathbb{Z}^d_{\vartheta^{-1}} \subset \mathbb{R}^d$, $1 \leq i \leq n$, in such a way that

$$x_i(\vartheta) = x_i + O(\vartheta^{-1}) \quad , \quad (\vartheta \to \infty) \ . \tag{13}$$

In particular we get

$$|x_i(\vartheta) - x_j(\vartheta)| = |x_i - x_j| + O(\vartheta^{-1}) \quad , \quad (\vartheta \to \infty) \ . \tag{14}$$

Then in the **scaling limit**, in which

$$\vartheta \to \infty \qquad , \tag{15}$$

the corresponding points $z_i = z_i(\vartheta) := \vartheta x_i(\vartheta) \in \mathbb{Z}^d$ satisfy (11).

General strategy, RG transformation

- *construct* from our *real* lattice system in some way, including the above scale transformation with scale parameter ϑ, an "effective" one.
- **First step:** *construction* of a **scaled lattice system**, "naturally" determined by the scale transformation (12).
- **Second step:** construction of an **effective** lattice system of $\mathbb{Z}^d$
- **RG transformation**: transformation mapping the original state of our real system to the effective state
- Crucial: **construction of the RG transformation** itsself, i.e. transf. actually **realized as a symmetry transformation.**
- Note: (scaled) "effective" lattice system only a *construction*; motivation: **extract the effective large scale behavior of the correlation functions** from our physical lattice system $\mathbb{Z}^d$.

Construction of a scaled lattice system

- Define the **scaled spin**, $\varphi^{(\vartheta)}$, on $\mathbb{Z}^d_{\vartheta^{-1}}$ as

$$\varphi^{(\vartheta)} : \mathbb{Z}^d_{\vartheta^{-1}} \to \mathbb{R}^d \quad , \quad x \mapsto \varphi^{(\vartheta)}(x) := \alpha(\vartheta)\, \varphi(\vartheta x) \ , \tag{16}$$

  where $\alpha(\vartheta)$ is a *positive* rescale factor.
- **scaled parameters** (w.r.t. $\mathbb{Z}^d_{\vartheta^{-1}}$) corresponding to just these (thermodynamic) parameters, which determine the **equilibrium state** of our real lattice system $\mathbb{Z}^d$. One such parameter is the **inverse temperature** β.

Construction of a scaled lattice system

- Want the *RG transformation* be realized as a *symmetry transformation*.
- Therefore we require, due to (2), the "**scaled inverse temperature**", $\beta(\vartheta)$, to satisfy

$$\beta(\vartheta) < \beta_c \qquad , \quad \forall \vartheta \in [1, \infty[\ . \tag{17}$$

Construction of a scaled lattice system

Intuitively expecting but also motivated by the fact that (free) energy has **scale dimension** 0 we set (with obvious notation):

$$H^{(\vartheta)}_{\vartheta^{-1}\Lambda}(\varphi^{(\vartheta)}) := H_{\Lambda}(\varphi) \qquad , \qquad d\lambda^{(\vartheta)}\left(\varphi^{(\vartheta)}(\vartheta^{-1}j)\right) := d\lambda(\varphi(j)) \quad , \; j \in \mathbb{Z}^d \; . \tag{18}$$

Construction of a scaled lattice system

- Define the **scaled equilibrium state** of $\mathbb{Z}^d_{\vartheta^{-1}}$, denoted by $d\mu^{(\vartheta)}_{\beta(\vartheta)}$,
- to be the **probability measure which solves the DLR-equations**, wherein **all expressions are replaced by their scaled counterparts**.
- **Corresponding (expectation value) functional:** $\langle\cdot\rangle^{(\vartheta)}_{\beta(\vartheta)}$.
- Note that due to (18) $d\mu^{(\vartheta)}_{\beta(\vartheta)}$ differs from $d\mu_\beta$ only in respect thereof that the **scaled parameters** have to be brought in, instead of the unscaled ones. Therefore

$$d\mu^{(\vartheta)}_{\beta(\vartheta)}(\varphi^{(\vartheta)}) = d\mu_{\beta(\vartheta)}(\varphi)\ . \tag{19}$$

Construction of a scaled lattice system

Now we can define the **scaled correlations** for points $y_i \in \mathbb{Z}^d_{\vartheta^{-1}}$ $(1 \leq i \leq n,\ \vartheta \in [1, \infty[)$:

$$G_\vartheta(y_1, \ldots, y_n) := \left\langle \varphi^{(\vartheta)}(y_1) \cdots \varphi^{(\vartheta)}(y_n) \right\rangle^{(\vartheta)}_{\beta(\vartheta)} \tag{20}$$

$$\overset{(16)}{\underset{(19)}{=}} \ \alpha(\vartheta)^n \ \langle \varphi(\vartheta y_1) \cdots \varphi(\vartheta y_n) \rangle_{\beta(\vartheta)} \tag{21}$$

(also called **rescaled correlation function**).

Renormalization condition

This finally enables us to state the **renormalization condition**:
For $x_1, \ldots, x_n \in \mathbb{Z}^d$:

$$0 < |x_i - x_j| < \infty\,,\ n \geq 2$$
$$\Rightarrow \quad 0 < \ G^*(x_1, \ldots, x_n) := \lim_{\vartheta \to \infty} G_\vartheta(x_1(\vartheta), \ldots, x_n(\vartheta)) \ < \infty \quad (22)$$

$G^*(x_1, \ldots, x_n)$ is called **limiting correlation function.**

General renormalization condition

- The above condition (22) is only the renormalization condition for a RG transformation which ends up with the "scaled" state $d\mu_{\beta(\vartheta)}$ (see (19)) as effective state of $\mathbb{Z}^d$.
- In general, where the **RG transformation ends up with just any effective state of** $\mathbb{Z}^d$, the **general renormalization condition** is obtained by considering the scaling limit of the **correlation function with respect to the effective state.**

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Summary: Consequences of the renormalization condition

- The renormalization condition yields a **connection to QFT**.
- The renormalization condition implies non-trivial **scaling relations** between critical exponents.
- **(determination of critical exponents)** An explicit construction of the renormalization condition determines, in principle, the critical exponents ν, γ and η.

A **second "renormalization condition"**

In order to get a scaling relation between the critical exponents η, ν and γ, it suffices to show a **second "renormalization condition"** (in addition to (22)):

$$\vartheta m(\beta(\vartheta)) \to m^* > 0 \quad , \quad \text{as } \vartheta \to \infty \, . \tag{23}$$

Consequences of the renormalization condition(s)

- **Scaled inverse temperature determined by ν :**

$$\boxed{\beta_c - \beta(\vartheta) \ \sim \ \vartheta^{-\frac{1}{\nu}} \quad , \quad \vartheta \to \infty} \, . \tag{24}$$

- **Scaling relation:**

$$\boxed{(2-\eta)\nu - \gamma = 0} \tag{25}$$

Determination of critical exponents

An explicit construction of the renormalization condition, i.e. having found a choice for the functions $\alpha(\vartheta)$ and $\beta(\vartheta)$, determines, in principle, the critical exponents ν, γ and η:

- $\alpha(\vartheta)$ determines η via (definition η):

$$\alpha(\vartheta)^2 \sim \vartheta^{d-2+\eta} \quad , \quad \vartheta \to \infty \, . \tag{26}$$

- $\beta(\vartheta)$ determines ν by (24), and
- γ is then obtained via the scaling relation (25).

Renormalization group transformations

- Here we only study the **static (time-averaged) case** but there exists a RG theory for *dynamics* as well.
- The ultimate ambition of RG theory is the **construction of a non-trivial limiting correlation function**, i.e. the **accomplishment of a renormalization condition**.
- a RG transformation is a combination of a **scale transformation** and some **coarse graining** process, leading finally to an effective state.

Typical example of a RG transformations: **RG block spin transformations.**

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RG block spin transformation

A RG block spin transformation consists of two transformations:

- a scale transformation and
- a block spin transformation.

The latter one is the reason why this kind of RG transformation is referred to as RG block spin transformation. As stated above, a scale transformation is somehow incorporated in every RG transformation.

Scale transformation

First we let act a **scale transformation** on our lattice (system) $\mathbb{Z}^d$. Since we are interested in the large scale behavior of our lattice system, we choose a scale transformation

$$\mathbb{Z}^d \to \mathbb{Z}^d_{\vartheta^{-1}} \quad , \quad z \mapsto \vartheta^{-1} z \tag{27}$$

where $\vartheta \in [1, \infty[$. (intuitively speaking: zooming out).
As a consequence of scaling the *lattice*, all the quantities which can be associated with our lattice get rescaled in some way (see last section). In particular we expect that the spin as well as all the thermodynamic parameters which determine the equilibrium state (especially the inverse temperature) become non-trivially rescaled. For the spin we therefore introduced the rescaling factor $\alpha(\vartheta)$ (see (16)); the scaled inverse temperature we denote by $\beta(\vartheta)$.

block spin transformation

Second we will do a **block spin transformation**, which is a (linear) transformation in spin (configuration) space $K_{\mathbb{Z}^d}$.
The construction is described below.

Block size

Characteristic to any block spin transformation is a number

$$L \in \mathbb{N} \quad : \quad L > 1 \quad , \tag{28}$$

called **block size.** For the following we fix it.

Main idea

The main idea is now to "divide" the scaled lattice $\mathbb{Z}^d_{\vartheta^{-1}}$, $\vartheta \in [1, \infty[$, into (hyper-) cubic blocks with side length $\epsilon := \vartheta^{-1} L$, then to take an average of all the spins contained in a block – this average one might call **block spin** – and identifying each block with a site in $\mathbb{Z}^d$; with this site is then associated the block spin.

Main idea - continued

Depending on how big ϑ was chosen in the scale transformation one might want to average over a bigger block with side length $\epsilon := \vartheta^{-1} L^m$. The question which then automatically arises is: How big do we want to have ϵ ? Since we want to identify each block with a site in $\mathbb{Z}^d$, a good answer is certainly $\epsilon \approx 1$. Therefore define

$$m \equiv m(\vartheta) := \min \{ j \in \mathbb{N} \ : \ \vartheta \leq L^j \} \tag{29}$$

and set the **scaled "block" size** to be

$$\epsilon := \vartheta^{-1} L^{m(\vartheta)} \, . \tag{30}$$

Hence

$$1 \leq \epsilon < L\,. \tag{31}$$

Note that due to (31)

$$\vartheta \to \infty \qquad \Leftrightarrow \qquad m \to \infty \quad . \tag{32}$$

When we use the latter as scaling limit we write ϑ_m instead of ϑ in order to remind of (32).

Construction of an effective lattice (system) $\mathbb{Z}^d$

"Effective": in the sense that we **extract only relevant information** from $\mathbb{Z}^d_{\vartheta^{-1}}$.

- Divide the lattice $\mathbb{Z}^d_{\vartheta^{-1}}$ into ϵ-blocks (blocks with side length ϵ) so that: ϵ-blocks centered in exactly these points which are contained in $C := \{y \in \mathbb{Z}^d_{\vartheta^{-1}} \mid \exists x \in \mathbb{Z}^d : y = \epsilon x\}$.
- Realizing that $\mathbb{Z}^d$ is naturally contained in $\mathbb{Z}^d_{L^{-m}}$ we do the following "identification map"

$$\mathbb{Z}^d_{\vartheta^{-1}} \to \mathbb{Z}^d_{L^{-m}} \quad , \quad y \mapsto \epsilon^{-1} y \, , \tag{33}$$

which maps any ϵ-block centered in $y := \epsilon x \in C \subset \mathbb{Z}^d_{\vartheta^{-1}}$, $x \in \mathbb{Z}^d$, to an 1-block centered in $x \in \mathbb{Z}^d \subset \mathbb{Z}^d_{L^{-m}}$.

Construction of an effective lattice (system) $\mathbb{Z}^d$

- Associate with each site $x \in \mathbb{Z}^d \subset \mathbb{Z}^d_{L^{-m}}$ a **block spin** $(r^{(\vartheta)}\varphi)(x)$, which is the average of all spins that are "contained" in the ϵ-block centered in $\epsilon x \in \mathbb{Z}^d_{\vartheta^{-1}}$.
- It follows the ultimate **block spin transformation**, a linear transformation in spin configuration space $K_{\mathbb{Z}^d}$, transforming the measure $\mu_{\beta(\vartheta)}$ into a measure $R^{(\vartheta)}\mu_{\beta(\vartheta)}$, which means intuitively establishing the block spin as a "real" spin on $\mathbb{Z}^d$.
- To what extent the measure $R^{(\vartheta)}\mu_{\beta(\vartheta)}$ really describes an ("effective") equilibrium state on $\mathbb{Z}^d$ will then be discussed.

After having constructed the block spin: "forget" the points contained in $\mathbb{Z}^d_{L^{-m}} \setminus \mathbb{Z}^d$. What we regard as relevant information has been absorbed into the block spin.

Now we want to introduce the notions of

- *block spin* and
- *block correlation function*

Block spin

Now define the **block spin**, $(r^{(\vartheta)}\varphi)(x)$, at site $x \in \mathbb{Z}^d$ to be

$$(r^{(\vartheta)}\varphi)(x) := \frac{1}{(L^m)^d} \sum_{\substack{y \,\in\, \mathbb{Z}^d_{\vartheta^{-1}} \\ \|\epsilon^{-1} y - x\|_\infty \le \frac{1}{2}}} \varphi^{\vartheta}(y) \tag{34a}$$

$$\overset{(16)}{=} \frac{1}{(L^m)^d} \sum_{\substack{y \,\in\, \mathbb{Z}^d_{\vartheta^{-1}} \\ \|\epsilon^{-1} y - x\|_\infty \le \frac{1}{2}}} \alpha(\vartheta)\, \varphi(\vartheta y)\,. \tag{34b}$$

The average is taken in dividing by $(L^m)^d = (\vartheta\epsilon)^d$ instead of ϵ^d. We do this because we get then the right average in the trivial case, where $\vartheta = 1$.

Block correlation function

At this point we recall why we want to construct a RG (block spin) transformation: we hope to accomplish the renormalization condition !
Therefore the following definition becomes important:
For $x_i \in \mathbb{Z}^d$, $1 \leq i \leq n$, define the **block correlation function**, $G_\vartheta(\kappa_{x_1}, \ldots, \kappa_{x_n})$, to be

$$G_\vartheta(\kappa_{x_1}, \ldots, \kappa_{x_n}) := \left(\frac{1}{(L^m)^d}\right)^n \sum_{\substack{y_1,\ldots,y_n \in \mathbb{Z}^d_{\vartheta^{-1}} \\ \|\epsilon^{-1} y_i - x_i\|_\infty \leq \frac{1}{2} \ , \ 1 \leq i \leq n}} G_\vartheta(y_1, \ldots, y_n) \,. \quad (35a)$$

$G_\vartheta(\kappa_{x_1}, \ldots, \kappa_{x_n})$ describes the *average correlation* between ϵ-blocks, centered in points $\epsilon x_i \in C \subset \mathbb{Z}^d_{\vartheta^{-1}}$, $1 \leq i \leq n$, with respect to the *scaled equilibrium state*.

Then:

$$
\begin{aligned}
G_\vartheta(\kappa_{x_1},\ldots,\kappa_{x_n}) &\overset{(35a)}{=} \left(\frac{1}{(L^m)^d}\right)^n \sum_{\substack{y_1,\ldots,y_n \in \mathbb{Z}^d_{\vartheta^{-1}} \\ \|\epsilon^{-1} y_i - x_i\|_\infty \le \frac{1}{2}\ ,\ 1\le i\le n}} G_\vartheta(y_1,\ldots,y_n) \\
&\overset{(20)}{=} \left(\frac{1}{(L^m)^d}\right)^n \sum_{\substack{y_1,\ldots,y_n \in \mathbb{Z}^d_{\vartheta^{-1}} \\ \|\epsilon^{-1} y_i - x_i\|_\infty \le \frac{1}{2}\ ,\ 1\le i\le n}} \alpha(\vartheta)^n \ \langle \varphi(\vartheta y_1)\cdots\varphi(\vartheta y_n)\rangle_{\beta(\vartheta)} \\
&\overset{(34b)}{=} \left\langle (r^{(\vartheta)}\varphi)(x_1)\cdots(r^{(\vartheta)}\varphi)(x_n)\right\rangle_{\beta(\vartheta)} \ . \qquad (36)
\end{aligned}
$$

Intuitively (37) says that the *average correlation* between the spins contained in ϵ-blocks, centered in points $\epsilon x_i \in C \subset \mathbb{Z}^d_{\vartheta^{-1}}$, $x_i \in \mathbb{Z}^d$, with respect to the *scaled equilibrium state* is just the *correlation* between the *block spins* in $x_i \in \mathbb{Z}^d$, $1 \le i \le n$, with respect to the state $d\mu_{\beta(\vartheta)}$ on $\mathbb{Z}^d$.

$$G_\vartheta(\kappa_{x_1},\ldots,\kappa_{x_n}) = \Big\langle (r^{(\vartheta)}\varphi)(x_1)\cdots(r^{(\vartheta)}\varphi)(x_n)\Big\rangle_{\beta(\vartheta)} \tag{37}$$

$$= \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} (r^{(\vartheta)}\varphi)(x_k)\; d\mu_{\beta(\vartheta)}(\varphi)\,. \tag{38}$$

Block spin transformation

Since $d\mu_{\beta(\vartheta)}$ is a translation-invariant (finite) probability measure on $K_{\mathbb{Z}^d}$ there is a unique (finite) probability measure $R^{(\vartheta)}\mu_{\beta(\vartheta)}$ on $K_{\mathbb{Z}^d}$ such that:

$$\int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} (r^{(\vartheta)}\varphi)(x_k)\, d\mu_{\beta(\vartheta)}(\varphi) = \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k)\, d(R^{(\vartheta)}\mu_{\beta(\vartheta)})(\varphi) \quad , \qquad (39)$$

for all $x_i \in \mathbb{Z}^d$, $(1 \leq i \leq n)$, and $n \in \mathbb{N} \setminus \{0\}$.

Checkpoint

Having defined the measure $R^{(\vartheta)}\mu_{\beta(\vartheta)}$, we have to check if

1. there is a lattice system (the 4 properties!) with an equilibrium state given by our new measure $R^{(\vartheta)}\mu_{\beta(\vartheta)}$.
2. $R^{(\vartheta)}\mu_{\beta(\vartheta)}$ satisfies the assumptions stated last section.

Important points: $R^{(\vartheta)}\mu_{\beta(\vartheta)}$ is

- invariant under lattice translations ? Yes!
- extremal invariant ? Yes!
- a Gibbs measure (i.e. a measure which satisfies the DLR-equations for some Hamilton function H) ? *Unfortunately*, this is *not true in general* !
  The proceeding is now to choose an *appropriate* space of Gibbs states which is *closed* under the action of any RG block spin transformation.

Renormalization condition

Finally from (38) and (39) we get

$$G^*(\kappa_{x_1}, \dots, \kappa_{x_n}) := \lim_{\vartheta \to \infty} G_\vartheta(\kappa_{x_1}, \dots, \kappa_{x_n})$$
$$= \lim_{\vartheta \to \infty} \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k) \; d(R^{(\vartheta)} \mu_{\beta(\vartheta)})(\varphi) \quad , \quad (40)$$

provided the limit exists.

Choosing the rescale factor for the spin: $\alpha(\vartheta)$

In order to arrive at an interesting concept we now suppose that $\alpha(\vartheta)$ is "proportional" to some power of ϑ,

$$\alpha(\vartheta)^2 \sim \vartheta^{d-2+\eta} \quad , \quad \vartheta \to \infty \qquad , \tag{41}$$

for some η (this is the *definition* of η).

Iteration: splitting $R^{(\vartheta)}$

"Small" block spin:

$$(r\varphi)(x) := L^{(\eta-d-2)/2} \sum_{\substack{z \,\in\, \mathbb{Z}^d \\ \|L^{-1}z-x\|_\infty \leq \frac{1}{2}}} \varphi(z)\ . \tag{42}$$

Connection: **block spin** $\longleftrightarrow$ **"Small" block spin**

$$(r^{(\vartheta)}\varphi)(x) = \alpha(\epsilon^{-1})\ (r^{m(\vartheta)}\varphi)(x)\ . \tag{43}$$

Analogous to (39) define $R\mu_{\beta(\vartheta)}$ to be the unique measure on $K_{\mathbb{Z}^d}$ such that:

$$\int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} (r\varphi)(x_k)\, d\mu_{\beta(\vartheta)}(\varphi) = \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k)\, d(R\mu_{\beta(\vartheta)})(\varphi) \quad , \tag{44}$$

for all $x_i \in \mathbb{Z}^d$, $(1 \leq i \leq n)$, and $n \in \mathbb{N} \setminus \{0\}$. Then

$$\int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} (r^{(\vartheta)}\varphi)(x_k)\, d\mu_{\beta(\vartheta)}(\varphi) = \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k)\, d(R^{m(\vartheta)}\mu_{\beta(\vartheta)})(\alpha(\epsilon)\varphi)\,. \tag{45}$$

Due to uniqueness we get

$$d(R^{(\vartheta)}\mu_{\beta(\vartheta)})(\varphi) = d(R^{m(\vartheta)}\mu_{\beta(\vartheta)})(\alpha(\epsilon)\varphi)\,. \tag{46}$$

Renormalization condition: iterated

So we can write (40) alternatively

$$G^*(\kappa_{x_1},\dots,\kappa_{x_n}) := \lim_{m\to\infty} G_{\vartheta_m}(\kappa_{x_1},\dots,\kappa_{x_n}) = \lim_{m\to\infty} \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k)\, d(R^m \mu_{\beta(\vartheta_m)})(\alpha(\epsilon)\varphi)\,. \quad (47)$$

Note that by (41), (42) and (52) the transformation $R \equiv R_\eta$ depends on the exponent η.

Outline

We start analyzing R_η

Let

- M be some appropriate chosen cone of finite measures, μ, on the space, $K_{\mathbb{Z}^d}$, of spin configurations and
- R_η be a RG block spin transformation, defined by (41), (42) and (52), *acting* on M ($R_\eta : M \to M$).
- Furthermore we assume that the action of R_η on M is smooth.

Fixed point

Then, a **fixed point** of R_η is a point $\mu^* \in M$ which is invariant under the action of R_η, i.e.

$$R_\eta \, \mu^* = \mu^* \; . \tag{48}$$

Obviously, if the scaling limit of a state $d\mu$,

$$d\mu^*(\varphi) := d(\lim_{m\to\infty} R^m \mu)(\varphi) \; , \tag{49}$$

exists, then $d\mu^*$ is a fixed point.

General strategy

Assume:

- we already know $\eta \longleftrightarrow \alpha(\vartheta)$ (corresponding to a specific class of spin systems)
- the existence of a fixed point of R_η.

Strategy: find *all* fixed points of R_η

Assume: *State* of a specific spin system will be **driven to** one of those *fixed points*.

- Found a fixed point? $\longrightarrow$ calculate the critical exponents, corresponding to this fixed point (Coming soon.)
- **Idea:** study how R_η behaves in the vicinity of a fixed point $\longrightarrow$ linearization of R_η at μ^*.

$\nu \longrightarrow \beta(\vartheta) \longrightarrow$ RG transformation is constructed

Stable, unstable manifolds

Now we *choose* a fixed point, μ^*, of R_η. We define

- $M_{f.p.} \equiv M_{f.p.}(R_\eta, \mu^*)$ to be the manifold of all fixed points of R_η passing through μ^*.

Under suitable conditions on M and R_η one can *decompose* M in the vicinity of μ^* into an **stable manifold**, $M_s(\mu^*)$, and an **unstable manifold**, $M_u(\mu^*)$, defined as follows:

- states on $M_s(\mu^*)$ are driven towards μ^* and
- states on $M_u(\mu^*)$ are driven away from μ^*.

"Relevant perturbations"

We are interested in the tangent spaces of these three manifolds at the point μ^*:

- $\mathcal{C}$ denotes the tangent space to $M_{f.p.}(R_\eta, \mu^*)$.
- The tangent space, $\mathcal{I}$, to $M_s(\mu^*)$, called space of "irrelevant pertubations", is the vector space spanned by eigenvectors of $DR_\eta(\mu^*)$ corresponding to eigenvalues of modulus < 1.
- The tangent space, $\mathcal{R}$, to $M_u(\mu^*)$, called space of "relevant pertubations", is the vector space spanned by eigenvectors of $DR_\eta(\mu^*)$ corresponding to eigenvalues of modulus > 1.

And then there is

- the tangent space, $\mathcal{M}$, at μ^*, called space of "marginal perturbations", which is the vector space spanned by eigenvectors of $DR_\eta(\mu^*)$ corresponding to eigenvalues of modulus 1.

Determination of the critical exponents ν and γ

The critical exponents ν and γ are determined by the spectrum of $DR_\eta(\mu^*)$, more precisely by the relevant eigenvalues (which have modulus > 1) of $DR_\eta(\mu^*)$.

Assumption tangent space

Assumption: The tangent space at μ^* splits into

- a *one-dimensional* space of *relevant* perturbations and
- a *co-dimension-one* space of *irrelevant* perturbations,

which means in particular that there are no further marginal perturbations. Assuming smoothness properties we get the starting point for the determination of the critical exponents:
For some neighborhood of μ^* there exist a one-dimensional unstable and a co-dimension-one stable manifold passing through μ^*.

The critical exponents

- space of relevant perturbations: 1-D
- $\implies$ $\exists$ unique simple relevant eigenvalue λ
- We get

$$\boxed{\nu = \frac{\log(L)}{\log(\lambda)}} \tag{50}$$

Independent of L !

- Via **scaling relation** we get the critical exponent γ.

Summary

- **Starting point:** At a critical point, the correlation length $\xi \equiv m^{-1}$ diverges.
- look at large scale behavior of correlation functions.
- scale transformation $\longrightarrow$ scaled lattice system

This leads to the construction of a RG transformation ending up with a **renormalization condition**: non-trivial existence of a **limiting correlation function. Consequences:**

- Scaling relations between critical exponents
- Determination of critical exponents
- Connection to QFT

Crucial: **construction** of a RG BS transformation

- Choice of $\alpha(\vartheta) \quad \longleftrightarrow \quad \eta$
- **"Small" block spin:**

$$(r\varphi)(x) := L^{(\eta-d-2)/2} \sum_{\substack{z \in \mathbb{Z}^d \\ \|L^{-1}z-x\|_\infty \leq \frac{1}{2}}} \varphi(z)\,. \tag{51}$$

- **"Small" block spin transformation:**

$$\int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} (r\varphi)(x_k)\, d\mu_{\beta(\vartheta)}(\varphi) = \int_{K_{\mathbb{Z}^d}} \prod_{k=1}^{n} \varphi(x_k)\, d(R\mu_{\beta(\vartheta)})(\varphi) \quad . \tag{52}$$

for all $x_i \in \mathbb{Z}^d$, $(1 \leq i \leq n)$, and $n \in \mathbb{N} \setminus \{0\}$.

Crucial: **construction** of a RG BS transformation

- Find fixed point of $R \equiv R_\eta$.
- Determination of ν by the relevant eigenvalues of the linearization of the RG transformation at a fixed point.

$$\boxed{\nu = \frac{\log(L)}{\log(\lambda)}} \tag{53}$$

- $\nu \longrightarrow \beta(\vartheta) \longrightarrow S^{(\vartheta)}$, where

$$S^{(\vartheta)}\mu_\beta := \mu_{\beta(\vartheta)} \tag{54}$$

- The RG BS transformation, $R^{(\vartheta)}S^{(\vartheta)}$, is constructed.

Connection to QFT

The limiting correlation functions of a classical lattice spin system may be the Euclidean Green's functions of a relativistic quantum field theory satisfying the Wightman axioms (W0)-(W5), i.e.

$$G^*(x_1, \ldots, x_n) \equiv S_n(x_1, \ldots, x_n) \tag{55}$$